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ON A POWER SERIES EXPANSION OF THE GRAVIATIONAL POTENTIAL OF AN INHOMOGENEOUS ELLIPSOID*

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An algorithm is given for successively computing the partial derivatives of the gravitational potential of an inhomogeneous ellipsoid, whereby the power expansion of this function can be constructed. The relevant recurrence procedures are proved. The coefficients of the Taylor series of the potential can be used when analysing the stability of points of libration. Ways of applying the results are indicated. The case of a homogeneous ellipsoid was considered in /1–3/.

1. The power function. Our aim below is to obtain an efficient algorithm for computing the coefficients of the power expansion of the Hamiltonian function in the problem of the motion of a point in the gravitational field of an inhomogeneous rotating ellipsoid. For this, we in fact need to compute the Taylor series for the gravitational potential.

Let $\delta : \mathbf{R}^3 \to \mathbf{R}_+$ be the density, specifying the mass distribution in the ellipsoid, where $\mathbf{R}_+ = \{x \in \mathbf{R}: x \geqslant 0\}$. It is such that $\delta: \mathbf{x} \to \delta_1(\mu)$, where μ numbers the ellipsoids of the one-parameter family of like elliposids:

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = \mu = \chi, \quad \chi \in [0, +\infty)$$

x_i $(i = 1, 2, 3)$ are coordinates in $\mathbf{R}^3$, and a_i $(i = 1, 2, 3)$ are the semi-axes of the chosen initial ellipsoid. The function δ specifies the finite Radon measure $\delta(\mathbf{x})\, d\mathbf{x}$, where $d\mathbf{x}$ is the ordinary Lebesgue measure in $\mathbf{R}^3$. The measure has to be finite, since the gravitating mass

$$\int_{\mathbf{R}^3} \delta(\mathbf{x})\, d\mathbf{x} = 4\pi a_1 a_2 a_3 \int_{\mathbf{R}_+} \delta_1(\chi^2)\chi^2\, d\chi = m < +\infty \tag{1.1}$$

is finite, or $\delta \in L_1(\mathbf{R}^3)$.

We can justify a study of this distribution because it approximately gives the density for certain ellipsoidal objects in the dynamics of star systems. The latter may be elliptic galaxies or the ellipsoidal centres of certain spiral galaxies.

The summability condition for δ leads to the summability of $\delta_1 \in L_1[\varepsilon, +\infty)$, since the function $\delta_1(\chi^2)\chi^2$ $(\varepsilon > 0)$ is integrable in the set $[\varepsilon, +\infty)$. Turning to the case of an interior point, described in /4/, we obtain the relation for the power function

$$U(\mathbf{x}) = f\pi a_1 a_2 a_3 \left[\int_0^{\mu_0} \delta_1(\mu) S(s)\, d\mu + \int_{\mu_0}^{+\infty} \delta_1(\mu) S(0)\, d\mu\right] \tag{1.2}$$

$$\mu_0 = \sum_{k=1}^{3} \frac{x_k^2}{a_k^2}$$

$$S(s) = \int_s^{+\infty} \frac{d\sigma}{R(\sigma)}, \quad R(\sigma) = [(a_1^2 + \sigma)(a_2^2 + \sigma)(a_3^2 + \sigma)]^{1/2}$$

Here, f is the gravitational constant and s satisfies the equation

$$\sum_{k=1}^{3} \frac{x_k^2}{(a_k^2 + s)} = \mu$$

The case of an exterior point in essence leads to the previous case if the density is interpreted as a distribution.

Let us show that the relations for the potential obtained in /4/ also hold when $\delta \in L_1(\mathbf{R}^3)$. In particular, we shall prove (1.2). The only thing that might stop us is divergence to zero with respect to the variable μ of the integral in (1.2).

We write

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$$U(\mathbf{x}) = U_\varepsilon(\mathbf{x}) + A_\varepsilon(\mathbf{x}), \quad A_\varepsilon(\mathbf{x}) = f \int_{D_\varepsilon} \frac{\delta(\mathbf{y})\, d\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$$

$$U_\varepsilon(\mathbf{x}) = f\pi a_1 a_2 a_3 \left[\int_\varepsilon^{\mu_0} \delta_1(\mu) S(s)\, d\mu + \int_{\mu_0}^{+\infty} \delta_1(\mu) S(0)\, d\mu\right]$$

where U_ε is the gravitational potential of the layers external to the ellipsoid D_ε, specified by the inequality $x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 \leqslant \varepsilon$, $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^3$.

For, if $\delta \in L_1(\mathbf{R}^3)$, then the Newtonian potential

$$U(\mathbf{x}) = \int_{\mathbf{R}^3} \frac{\delta(\mathbf{y})\, d\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$$

is a locally Lebesgue-integrable function (see e.g., /5, p.24/), defined almost everywhere. Since a Lebesgue integral is an additive set function, our expansion of U holds by virtue of the division $\mathbf{R}^3 = (\mathbf{R}^3 \setminus D_\varepsilon) \cup D_\varepsilon$. The integral as a set function has the property of absolute continuity (/6, p.282/). Hence $A_\varepsilon(\mathbf{x}) \to 0$ as $\varepsilon \to 0$, and finally $U_\varepsilon(\mathbf{x}) \to U(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^3 \setminus \{0\}$.

We specify the absolutely continuous function

$$\varkappa(\mu) = \int_\mu^{+\infty} \delta_1(\sigma)\, d\sigma$$

It is well defined, since $\delta_1 \in L_1[\mu, +\infty)$ for any $\mu > 0$. Using the extension to the case of a Lebesgue integral of the formula for integration by parts (/7/,p292/) in the limits $\varepsilon \leqslant \mu \leqslant +\infty$, and apssing in this formula to the limit as $\varepsilon \to 0$, we obtain

$$\int_0^{\mu_0} \delta_1(\mu) S(s)\, d\mu = -\varkappa(\mu_0) S(0) + \int_0^{+\infty} \varkappa(\mu) \frac{ds}{R(s)}$$

if we have the condition $\varkappa(\mu) S(s) \to 0$ as $\mu \to 0$. It will automatically be satisfied e.g., if $\varkappa(\mu)$ has a limit as $\mu \to 0$, which is true in most applications. Hence we can write the potential as

$$U(\mathbf{x}) = f\pi a_1 a_2 a_3 \int_0^{+\infty} \varkappa(\mu) \frac{ds}{R(s)}$$

We know from potential theory that, if $\delta \in C^\infty(G)$ in the domain $G \subset \mathbf{R}^3$, then $U \in C^\infty(G)$. The same is true for analyticity in G. In the case of an ellipsoidal mass distribution, smoothness of δ_1 in an interval leads to smoothness of U in the corresponding ellipsoidal layer.

2. **Fundamental theorems**. The equations of motion of a star in the coordinate system rotating with angular velocity ω along with the galaxy about its principal central axis of inertia are

$$x_1^{\cdot\cdot} - 2\omega x_2^{\cdot} - \omega^2 x_1 = U_{x_1}, \quad x_2^{\cdot\cdot} + 2\omega x_1^{\cdot} - \omega^2 x_2 = U_{x_2}, \quad x_3^{\cdot\cdot} = U_{x_3}$$

We transform to dimensionless variables in accordance with the relations $t = \tau/\omega$, $x_i = l\xi_i$ $(i = 1, 2, 3)$, $l^2 = a_1^2 + a_2^2 + a_3^2$ and obtain the new equations of motion

$$\xi_1'' - 2\xi_2' - \xi_1 = \Lambda_{\xi_1}, \quad \xi_2'' + 2\xi_1' - \xi_2 = \Lambda_{\xi_2}, \quad \xi_3'' = \Lambda_{\xi_3}$$

The prime denotes differentiation with respect to the new independent variable τ. In dimensionless variables the power function is

$$\Lambda(\xi) = \rho \int_0^{+\infty} h(\mu) \frac{du}{Q(u)}, \quad \mu = \sum_{k=1}^{3} \frac{\xi_k^2}{\alpha_k + u} \tag{2.1}$$

$$\rho = \frac{3fm}{4\omega^2 l^3}, \quad h(\mu) = \int_\mu^{+\infty} \gamma(v)\, dv, \quad \int_0^{+\infty} \gamma(u^{2/3})\, du = 1$$

$$\gamma(\mu) = \frac{4\pi a_1 a_2 a_3}{3m} \delta_1(\mu)$$

$$Q(u) = [(\alpha_1 + u)(\alpha_2 + u)(\alpha_3 + u)]^{1/2}, \quad a_i^2 = l^2 \alpha_i \ (i = 1, 2, 3)$$

If we perform a Legendre transformation and pass to the Hamiltonian system

$$\xi' = H_\eta, \quad \eta' = -H_\xi, \quad \xi, \eta \in \mathbf{R}^3 \tag{2.2}$$

the Hamiltonian becomes

$$H(\xi, \eta) = \|\eta\|^2/2 + (\eta_1\xi_2 - \eta_2\xi_1) - \Lambda(\xi)$$

To study the local behaviour of this system in the neighbourhood of a position of equilibrium, we need to know the power series expansion of H. For this, we have to compute the expansion of the function Λ. This means in practice that we must be able to calculate the partial derivatives of Λ of the requisite order.

It will be assumed throughout that the function $\varkappa$ (or h) is bounded, as we assumed when obtaining the expression for the potential. When evaluating the partial derivatives of $\Lambda(\xi)$, we require differentiation of the function I: $\mathbf{R}^3 \setminus \{0\} \to \mathbf{R}$ given by

$$I(\xi) = \int\limits_0^{+\infty} F(u, \xi) h[\mu(u, \xi)]\,du \tag{2.3}$$

where h and μ are given by (2.1). The function F: $\mathbf{R}_+ \times (\mathbf{R}^3 \setminus \{0\}) \to \mathbf{R}$ is analytic and is given by

$$F(u, \xi) = \frac{P_1^0(\xi)u^n + \ldots + P_1^n(\xi)}{P_2^0(\xi)u^n + \ldots + P_2^n(\xi)} \cdot \frac{\rho}{Q(u)}, \quad n \in \mathbf{N} \tag{2.4}$$

where $P_i^k(\xi)$ $(i = 1, 2;\ k = 0, \ldots, n)$ are polynomials in ξ_1, ξ_2, ξ_3, and all the $P_2^k(\xi)$ depend only on ξ_i^2 $(i = 1, 2, 3)$ and have positive coefficients. Moreover, $\deg P_2^0 > 1$ and with $\xi \in \mathbf{R}^3 \setminus \{0\}$, $P_2^0(\xi) > 0$ (it is positive definite).

Evalutation of thenext derivative of higher order implies obtaining integrals of type (2.3) with integrands of type (2.4), in which all the above properties of polynomials $P_i^k(\xi)$ $(i = 1, 2;\ k = 0, \ldots, n)$ are satisfied. At the start of this process, starting from (2.1), we must put $F(u, \xi) = F_0(u) = \rho/Q(u)$. The functions (2.4) with these properties form a module M over the ring of polynomials $\mathbf{R}[\xi_1, \xi_2, \xi_3]$ of three variables with real coefficients.

Theorem 1. If $\gamma \in L_1(\mathbf{R}_+)$ and $F \in M$, then $I \in C^1(\mathbf{R}^3 \setminus \{0\})$ and we have

$$I_{\xi_i} = \int\limits_{\mathbf{R}_+} F_{\xi_i}(u, \xi) h[\mu(u, \xi)]\,du - 2\xi_i \int\limits_{\mathbf{R}_+} F(u, \xi) \frac{\gamma[\mu(u, \xi)]}{\alpha_i + u}\,du \tag{2.5}$$

$$I_{\xi_i} = 2\xi_i F(0, \xi) h[\mu_0(\xi)] \Big(\alpha_i \sum_{k=1}^{3} \frac{\xi_k^2}{\alpha_k^2}\Big)^{-1} + \int\limits_{\mathbf{R}_+} G(u, \xi) h[\mu(u, \xi)]\,du \tag{2.6}$$

$$G(u, \xi) = F_{\xi_i}(u, \xi) + \frac{\partial}{\partial u}\Big\{2\xi_i F(u, \xi) \Big[(\alpha_i + u) \sum_{k=1}^{3} \frac{\xi_k^2}{(\alpha_k + u)^2}\Big]^{-1}\Big\} \tag{2.7}$$

and also

$$G(u, \xi) = \frac{Q_1^0(\xi)u^m + \ldots + Q_1^m(\xi)}{Q_2^0(\xi)u^m + \ldots + Q_2^m(\xi)} \cdot \frac{\rho}{Q(u)}, \quad m \in \mathbf{N},\ G \in M \tag{2.8}$$

Proof. The proof is quite laborious. The outline and main ideas are as follows. It is all a question of justifying differentiation with respect to the variables ξ_i $(i = 1, 2, 3)$, as with respect to the parameters, under the improper integral sign of (2.3). For this, we write (2.3) as

$$I(\xi) = -\int\limits_0^{\mu_0} F[u(\mu, \xi), \xi] \frac{\partial u}{\partial \mu}(\mu, \xi) h(\mu)\,d\mu\,,$$

where the variables of integration u and μ are assumed to be connected by the second equation of (2.1).

We next consider the result of a formal differentiation of (2.3) with respect to ξ_i:

$$I_{\xi_i}(\xi) = -\int\limits_0^{\mu_0} [(F_u u_{\xi_i} + F_{\xi_i}) u_\mu + F u_{\mu\xi_i}] h\,d\mu - \mu_{\xi_i} F u_\mu h\,|_{\mu=\mu_0} \tag{2.9}$$

By means of estimates for the factors and terms of the integrand, we can obtain the inequality

$$|[(F_u u_{\xi_i} + F_{\xi_i}) u_\mu + F u_{\mu\xi_i}] h| \leqslant c\mu^{-1/2}\,,$$

which holds finally as $\mu \to 0$ uniformly in any sufficiently small neighbourhood V of any point $\xi \in \mathbf{R}^3 \setminus \{0\}$ (V must have a compact closure). Theimproper integral in (2.9) is thus locally uniformly convergent and, by a well-known theorem of analysis (/8/, p.794), the differentiation formula (2.9) is valid.

The proof of (2.6), (2.7) is obtained by direct evaluations in (2.9), using integration by parts. The fact that $G \in M$ (see (2.8)) is proved by noting that both terms on the right-hand side of (2.7), and hence their sum G, belongs to module M.

Relations (2.6) and (2.7) enable us to evaluate any partial derivatives of the function Λ, using a standard recurrence procedure, provided, of course, that in domain $V \subset \mathbf{R}^3 \setminus \{0\}$ derivatives of sufficiently high order of the function $\gamma \circ \mu_0$ exist (in the ellipsoidal layer covering the domain V).

Corollary. Under the conditions of Theorem 1, the potential Λ of the ellipsoidal distribution have in $\mathbf{R}^3 \setminus \{0\}$ continuous partial derivatives of first order, and second-order derivatives which are defined almost everywhere.

Proof. For clearly, in view of (2.8), the integral in (2.6) (the second term on the right-hand side) converges uniformly in a fairly small neighbourhood V of any fixed point $\xi \in \mathbf{R}^3 \setminus \{0\}$, and hence is a continuous function in $\mathbf{R}^3 \setminus \{0\}$. The first term in (2.6) is likewise a continuous function in $\mathbf{R}^3 \setminus \{0\}$, since h is continuous. With regard to the second derivatives, the same Theorem 1 guarantees continuous derivatives of the second term of (2.6). For the first term, however, the derivatives are defined almost everywhere, since the function h is absolutely continuous.

If is often more convenient to have expressions for the derivatives directly in terms of density γ. For this, we use.

Theorem 2. If, in a neighbourhood V of the point $\xi^0 \in \mathbf{R}^3 \setminus \{0\}$, the function $\gamma \circ \mu_0 \in C^p(V), p \in \mathbf{N}$, while $\gamma \in L_1(\mathbf{R}_+)$, then in this neighbourhood $\Lambda \in C^{p+2}(V)$, and given any integer k such that $2 \leqslant k \leqslant p+2$, we have

$$D^{\mathbf{k}}\Lambda(\xi) = \sum_{i=0}^{k-2} R_i(\xi)\gamma^{(i)}[\mu_0(\xi)] + \int_{\mathbf{R}_+} F(u, \xi)\gamma[\mu(u, \xi)]\,du, \quad \gamma^{(i)} = \frac{d^i\gamma}{d\mu^i}, \quad \xi \in V \tag{2.10}$$

where R_i are rational functions, defined in $\mathbf{R}^3 \setminus \{0\}$, while the function $F \in M$, and in its representation (2.4) we have to put $P_1^0(\xi) \equiv 0$, and $D^{\mathbf{k}}$ is the differential operator

$$D^{\mathbf{k}} = \frac{\partial^k}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}\partial\xi_3^{k_3}}, \quad k_1 + k_2 + k_3 = k, \quad \mathbf{k} = (k_1, k_2, k_3)$$

Proof. We use induction on k, starting with $k = 2$.
By Theorem 1, we can obtain by direct calculations in V:

$$\Lambda_{\xi_i}(\xi) = -2\xi_i\rho \int_{\mathbf{R}_+} \frac{\gamma[\mu(u, \xi)]}{(\alpha_i + u)Q(u)}\,du \quad (i = 1, 2, 3) \tag{2.11}$$

and for the second-order derivatives:

$$\Lambda_{\xi_i\xi_j}(\xi) = -4\rho\xi_i\xi_j\gamma[\mu_0(\xi)]\left[\alpha_i\alpha_j(\alpha_1\alpha_2\alpha_3)^{1/2}\sum_{l=1}^{3}\frac{\xi_l^2}{\alpha_l^2}\right]^{-1} + \int_{\mathbf{R}_+} F(u, \xi)\gamma[\mu(u, \xi)]\,du \quad (i, j = 1, 2, 3) \tag{2.12}$$

$$F(u, \xi) = -\frac{2\xi_j}{\alpha_j + u}\frac{\partial}{\partial u}F_1(u, \xi) - \left[\sum_{l=1}^{3}\frac{\xi_l^2}{(\alpha_l + u)^2}\right]\frac{\partial}{\partial\xi_j}F_1(u, \xi)$$

$$F_1(u, \xi) = 2\rho\xi_i\left[(\alpha_i + u)Q(u)\sum_{l=1}^{3}\frac{\xi_l^2}{(\alpha_l + u)^2}\right]^{-1}$$

Clearly, $F(u, \xi)$ has the form required in the theorem and has the necessary properties.

If we make the inductive assumption that (2.10) holds for some $k \geqslant 2$, then, by using Theorem 1 for the integral appearing in (2.10) and performing the appropriate calculations, we obtain the relations

$$\frac{\partial}{\partial\xi_i}[D^{\mathbf{k}}\Lambda(\xi)] = 2\xi_i F(0, \xi)\gamma[\mu_0(\xi)]\left(\alpha_i\sum_{l=1}^{3}\frac{\xi_l^2}{\alpha_l^2}\right)^{-1} + \int_{\mathbf{R}_+} G(u, \xi)\gamma[\mu(u, \xi)]\,du + \frac{\partial}{\partial\xi_i}\left\{\sum_{j=0}^{k-2}R_j(\xi)\gamma^{(j)}[\mu_0(\xi)]\right\} \tag{2.13}$$

$$G(u, \xi) = \frac{2\xi_i}{\alpha_i + u}\frac{\partial}{\partial u}G_1(u, \xi) + \left[\sum_{l=1}^{3}\frac{\xi_l^2}{(\alpha_l + u)^2}\right]\frac{\partial}{\partial\xi_i}G_1(u, \xi)$$

$$G_1(u, \xi) = F(u, \xi)\left[\sum_{l=1}^{3}\frac{\xi_l^2}{(\alpha_l + u)^2}\right]^{-1} \quad (i = 1, 2, 3)$$

We can reduce (2.13) to the form (2.10):

$$\frac{\partial}{\partial \xi_i}[D^k\Lambda(\xi)] = \sum_{j=0}^{k-1} Q_j(\xi)\,\gamma^{(j)}[\mu_0(\xi)] + \int_{\mathbf{R}_+} G(u,\ \xi)\,\gamma[\mu(u,\ \xi)]\,du$$
$$(i = 1,\ 2,\ 3)$$

where $Q_j(\xi)$ are rational functions, while $G(u, \xi)$, like $F(u, \xi)$, satisfies the conditions of the theorem.

Using (2.13), we can computer-evaluate automatically the coefficients of the power expansion of the Hamiltonian function H in the neighbourhood of an equilibrium position, up to any required order, provided, of course, that γ is suitably smooth. In combination with methods for the automatic evaulation of normal forms H (e.g., the Depris-Hory method*, (*Markeev A.P. and Sokol'skii A.G., Some computational algorithms for the normalization of Hamiltonian systems, Preprint In-ta prikl. matem. Akad. Nauk SSSR, Moscow, 31, 1976.) we can obtain a method for a numerical-analytic study of the equilibrium positions of the problem mentioned at the start of Sect.2.

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APPROXIMATE SOLUTION OF SOME PERTURBED BOUNDARY VALUE PROBLEMS*

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A perturbation method for solving some linear boundary-value eigenvalue and eigenfunction problems is developed and justified. The class of problem considered is frequently encountered in applications when investigating elastic oscillatory systems with distributed and slightly variable parameters (a string, an elastic shaft, a beam, etc.), described by boundary value problems for hyperbolic-type equations with variable coefficients. A procedure for the approximate solution of these problems is developed with the required degree of accuracy with respect to the small parameter characterising the non-homogeneity. In particular, Dirichlet's problem, describing the oscillations of non-homogeneous elastic systems with clamped ends, is considered.

1. Formulation of the problem. The eigenvalue and eigenfunction problem for a linear perturbed second-order equation is considered in the real domain:

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